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# Attenuation and focusing of electromagnetic surface waves rounding gentle bends

M V Berry

H H Wills Physics Laboratory, Tyndall Avenue, Bristol BS8 1TL, UK

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**Abstract.** We study electromagnetic waves in and near dielectric surfaces  $S$  whose radii of curvature  $R$  are large compared with the surface wavelength  $2\pi/K_z$ , for cases when the dielectric constant  $\epsilon < -1$ . If such gentle bends are convex towards the vacuum the wave is not perfectly bound—it attenuates with decay length  $K_z^{-1}$ , the lost energy being radiated to infinity. The radiation appears to emerge tangentially into vacuum from the height  $z_r$  above  $S$  where the wave changes from evanescent to oscillatory. We obtain analytic formulae for  $z_r$  and  $K_z$  in terms of  $K_z R$  and  $\epsilon$ . When  $S$  is concave to the vacuum we argue that there should be no attenuation. On general gently bent surfaces the wave energy travels along rays that are geodesics on  $S$ . We discuss the focusing of families of rays on  $S$  and show that the imperfect focus of a plane wave incident on a general circularly symmetric hill is a cusped caustic with two rainbows as asymptotes. Perfect focusing is also possible and we calculate the shape of the 'geodesic lens' that would produce this. Finally, we suggest some experiments to test the theory.

## 1. Introduction

A flat interface between vacuum and a material with negative dielectric constant can support bound electromagnetic waves which travel along the surface and decay in both directions away from it. These electromagnetic surface waves are often called 'surface plasma oscillations' or, when quantized, 'surface plasmons'. They have been observed on the surfaces of metals and semiconductors, and are the subject of an extensive literature which is summarized in excellent reviews by Economou and Ngai (1974), hereafter called I, and Kliewer and Fuchs (1974), hereafter called II.

When the interface is not flat, the surface wave is no longer bound—it radiates into the vacuum, and may be excited by radiation incident from the vacuum. The literature (see I and II) contains theories of waves on surfaces which are cylindrical, spherical, weakly periodically rough (gratings) and weakly randomly rough (see also Celli *et al* (1975) and Marvin *et al* (1975)). However, there is no general discussion of *gently bent surfaces*. By 'gently bent' we mean that the radii of curvature are large in comparison with the vacuum wavelength  $\lambda$ , so that the surface is sufficiently locally flat for the surface wave to be almost perfectly bound. Over great distances such surfaces may however depart arbitrarily far from any chosen plane and so cannot be considered as perturbations of a plane.

In this paper we generalize results implicit in previous theoretical treatments (I) of cylindrical surfaces and make predictions and conjectures about surface waves on general gentle bends. We show that the effect of curvature is to cause the surface wave to *attenuate* in a way that depends on the local curvature, by a mechanism that is totally

different both from the diffuse scattering caused by small-scale random roughness and from Joule heating. We obtain formulae for the attenuation coefficient as a function of frequency and curvature and suggest experiments to test the correctness of these formulae. In addition, we discuss the focal structure of surface waves refracted by smooth surface irregularities ('hills').

First it is necessary to outline the theory leading to the basic equations. We seek electromagnetic fields  $\mathcal{E}$ ,  $\mathcal{D}$ ,  $\mathcal{B}$ ,  $\mathcal{H}$ , current density  $\mathcal{J}$  and charge density  $\varrho$ , all these quantities being position- and time-dependent. Only wavelengths  $\lambda$  greatly exceeding atomic dimensions will be considered; therefore the correct theoretical framework is provided by Maxwell's continuum equations and the conservation of charge, namely

$$\begin{aligned}\nabla \times \mathcal{E} &= -\frac{\partial \mathcal{B}}{\partial t} & \nabla \cdot \mathcal{D} &= \varrho \\ \nabla \times \mathcal{H} &= \mathcal{J} + \frac{\partial \mathcal{D}}{\partial t} & \nabla \cdot \mathcal{B} &= 0 \\ \nabla \cdot \mathcal{J} &= -\frac{\partial \varrho}{\partial t}.\end{aligned}\quad (1)$$

For monochromatic fields with frequency  $\omega (= 2\pi c/\lambda)$  we introduce complex frequency-dependent fields, currents and charges by

$$\{\mathcal{E}, \mathcal{D}, \mathcal{B}, \mathcal{H}, \mathcal{J}, \varrho\} \equiv \text{Re } e^{-i\omega t} \{E, D, B, H, J, \rho\}.\quad (2)$$

If the medium is non-magnetic and has a response which is linear and local, then it can be specified by its conductivity  $\sigma(\omega)$  and relative permittivity  $\epsilon_r(\omega)$  and the fields are related by

$$\mathbf{J} = \sigma \mathbf{E} \quad \mathbf{D} = \epsilon_0 \epsilon_r \mathbf{E} \quad \mathbf{B} = \mu_0 \mathbf{H}\quad (3)$$

where  $\epsilon_0$  and  $\mu_0$  are respectively the vacuum permittivity and permeability. From equations (1), (2) and (3) it is an elementary exercise to derive the following equation for  $\mathbf{B}$  in each homogeneous region:

$$\nabla^2 \mathbf{B} + k^2 \epsilon \mathbf{B} = 0,\quad (4)$$

where  $k \equiv \omega/c$  is the vacuum wavenumber and  $\epsilon(\omega)$  is the *complex dielectric constant*, defined as

$$\epsilon(\omega) \equiv \epsilon_r(\omega) + \frac{i\sigma(\omega)}{\epsilon_0 \omega}.\quad (5)$$

In vacuo,  $\epsilon(\omega) = 1$ .

Surface waves arise when  $\epsilon(\omega)$  is approximately real and when  $\text{Re } \epsilon(\omega) < -1$ . These conditions can be satisfied in metals and semiconductors. In a metal, free electron theory (see I) gives the approximation

$$\epsilon(\omega) \approx 1 - \frac{\omega_p^2}{\omega^2 [1 + (i/\omega\tau)]}\quad (6)$$

where  $\omega_p$  is the plasma frequency (typically  $10^{16} \text{ s}^{-1}$ ) and  $\tau$  is the relaxation time

(typically  $10^{-9}$  s). Since  $\omega_p\tau$  is very large, it is possible to choose  $\omega$  in the range

$$\frac{\omega_p}{\sqrt{2}} > \omega \gg \frac{1}{\tau} \tag{7}$$

and equation (6) shows that  $\epsilon(\omega)$  is then indeed real and less than  $-1$ .

Now we specialize to the two-dimensional problem where the surface curves in any  $xz$  plane but not along  $y$ , and where the fields are independent of  $y$  (we shall consider waves on more general surfaces in §§ 5 and 6). In this case surface waves only exist for the polarization corresponding to  $\mathbf{B}$  pointing along the direction  $\mathbf{j}$  of the  $y$  axis. This is called ‘p polarization’ (see I). Thus we write

$$\mathbf{B} = \mathbf{j}B(x, z, \omega) \tag{8}$$

and of course equation (4) holds with  $\mathbf{B}$  replaced by  $B$ . The boundary conditions on  $B$  across the medium–vacuum interface  $S$  can be found from the integral forms of equation (1), using equations (2) and (3). We obtain

$$B \quad \text{and} \quad \frac{1}{\epsilon}(\nabla B)_{\perp S} \tag{9}$$

continuous across  $S$ . Finally, we shall require the Poynting vector  $\mathbf{\Pi}$ , which gives the areal power density. In terms of the complex scalar wavefunction  $B$ ,  $\mathbf{\Pi}$  is given by

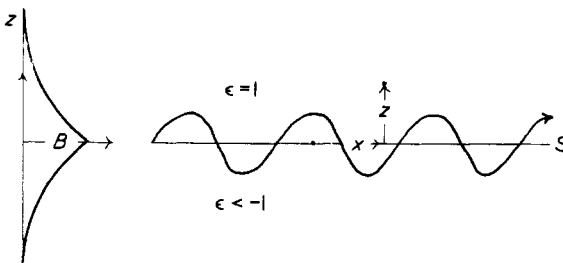
$$\mathbf{\Pi} \equiv (\mathbf{E} \times \mathbf{H})_{\text{time average}} = \frac{c^2}{2\mu_0\omega} \text{Im} \left( \frac{\mathbf{B}^* \nabla B}{\epsilon} \right). \tag{10}$$

### 2. Flat surfaces

We work briefly through this well-known case in order to establish results that will be used subsequently for curved surfaces. Let the flat surface  $S$  correspond to  $z = 0$ , with  $z > 0$  vacuum with  $\epsilon = 1$  and  $z < 0$  dielectric medium with  $\epsilon < -1$ . Let us seek a surface wave travelling along  $+x$  with ‘surface wavenumber’  $K(\omega)$ , and decaying as  $|z| \rightarrow \infty$ . Inspection of equation (4) (with (8)) shows that such a wave is indeed possible, and has the form (figure 1)

$$\begin{aligned} B(x, z) &= \exp(iKx) \exp[-z(K^2 - k^2)^{1/2}] && (z > 0) \\ &= \exp(iKx) \exp[+z(K^2 + |\epsilon|k^2)^{1/2}] && (z < 0) \end{aligned} \tag{11}$$

where we have normalized  $|B|$  to be unity on  $S$ .



**Figure 1.** Bound wave on a flat surface.

The formulae (11) satisfy the wave equation and the first of the boundary conditions (9). To satisfy the second we must match  $(\partial B/\partial z)/\epsilon$  across  $S$ . This constrains the surface wavenumber to be

$$K(\omega) = \frac{\omega}{c} \left( \frac{\epsilon(\omega)}{\epsilon(\omega) + 1} \right)^{1/2} = k \left( \frac{|\epsilon|}{|\epsilon| - 1} \right)^{1/2}. \tag{12}$$

This ‘dispersion relation’ shows that the surface wave propagates along  $x$  without attenuation ( $K$  real) if  $\epsilon(\omega) < -1$ , and that we then have  $K > k$  so that the wave (11) is indeed evanescent (and not propagating along  $z$ ) for  $z > 0$  (for  $z < 0$  waves are evanescent for all real  $K$ ).

The Poynting vector (equation (10)) for the wave (11) is directed along the  $x$  axis. The total power flowing is  $P$  where

$$\begin{aligned} P &\equiv \int_{-\infty}^{\infty} dz \Gamma(z) = \frac{c^2}{2\mu_0\omega} \int_{-\infty}^{\infty} dz \frac{B^*}{\epsilon} \frac{\partial B}{\partial z} \\ &= \frac{c^2}{2\mu_0\omega} \left( \int_0^{\infty} dz B^* \frac{\partial B}{\partial z} + \frac{1}{\epsilon} \int_{-\infty}^0 dz B^* \frac{\partial B}{\partial z} \right) \\ &= \frac{c^2(|\epsilon|^2 - 1)}{4\mu_0\omega|\epsilon|^{3/2}}. \end{aligned} \tag{13}$$

In the last step we have used equations (11) and (12). Finally, we remark that a surface wave would not exist if  $E$  rather than  $B$  pointed along  $j$  (‘s polarization’); for then  $\nabla E$  rather than  $\nabla B/\epsilon$  would be continuous across  $S$  and no wave of the form (11) can satisfy this condition.

### 3. Cylindrical bends—exact solution

A gently bent surface is locally flat, that is, its slope is locally constant. Therefore we expect locally at least to be able to find surface waves of the kind just discussed (equations (11)–(13)). To the next approximation, the curvature of a gently bent surface is locally constant, so that any local modification of the propagation of the surface waves can be found (in the two-dimensional cases we are presently considering) by studying propagation around bends which are portions of circular cylinders.

Therefore we consider surface waves on a cylinder of radius  $R$ , with vacuum outside and dielectric inside. In cylindrical polar coordinates  $r, \phi$  the wave equation (4) is separable, and the family of solutions specified by the separation constant  $a$  is

$$\begin{aligned} B(r, \phi) &\propto e^{ia\phi} H_a^{(1)}(kr) \quad \text{or} \quad e^{ia\phi} H_a^{(2)}(kr) & (r > R) \\ &\quad \text{(outgoing)} \quad \quad \quad \text{(incoming)} \\ &\propto e^{ia\phi} I_a(kr\sqrt{|\epsilon|}) & (r < R) \end{aligned} \tag{14}$$

where  $H_a^{(1)}$ ,  $H_a^{(2)}$  and  $I_a$  denote standard Hankel and modified Bessel functions (Abramowitz and Stegun 1964, ch 9). The solution  $I_a$  for  $r < R$  decays away from  $S$  towards the origin  $r = 0$ , as required for a surface wave. For  $r \gg R$  both  $H_a^{(1)}$  and  $H_a^{(2)}$ , as well as all combinations formed from these, are oscillatory rather than decaying functions of  $kr$ . This tells us that surface waves completely damped away from  $S$  cannot exist: they must radiate and/or be fed by incident radiation. We choose the *outgoing*

solution  $H_a^{(1)}$  to describe purely radiating waves. We do not constrain  $a$  to be an integer, because we are not considering complete cylinders but only locally cylindrical bends.

Next we put the solution (14) into closer correspondence with the flat surface solution (11) by defining new coordinates  $z, \xi$ , and a new parameter  $K$  instead of  $a$ , by

$$z \equiv r - R \quad \xi \equiv R\phi \quad K \equiv a/R \tag{15}$$

(figure 2), and normalizing  $|B|$  to be unity on  $S$ . This gives

$$\begin{aligned} B(\xi, z) &= e^{iK\xi} H_{KR}^{(1)}(kR + kz) [H_{KR}^{(1)}(kR)]^{-1} & z > 0 \\ &= e^{iK\xi} I_{KR}[k\sqrt{|\epsilon|(R+z)}] [I_{KR}(k\sqrt{|\epsilon|R})]^{-1} & z < 0. \end{aligned} \tag{16}$$

From equations (15) we see that  $\xi$  measures distance on  $S$ , ie when  $z = 0$ , but corresponds to a linearly scaled distance when  $z \neq 0$ . Therefore  $K$  has the same meaning in equation (16) as it does in equation (11): it is the surface wavenumber.

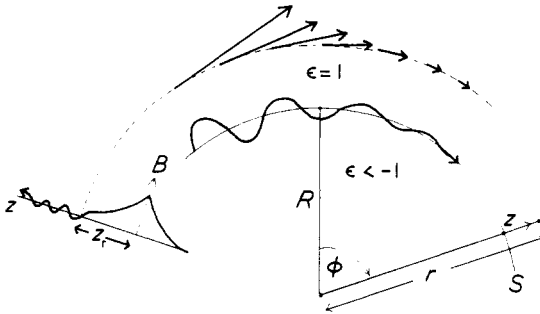


Figure 2. Quasi-bound wave on a cylindrical bend.

The formulae (16) satisfy the wave equation and the first of the boundary conditions (9). To satisfy the second we must match  $(\partial B/\partial z)/\epsilon$  across  $S$ . This gives the following equation that must be satisfied by  $K$  for the wave with frequency  $\omega$ :

$$-\sqrt{|\epsilon|} \frac{I_{KR}(k\sqrt{|\epsilon|R})H_{KR}^{(1)\prime}(kR)}{I_{KR}(k\sqrt{|\epsilon|R})H_{KR}^{(1)}(kR)} = 1 \tag{17}$$

where primes denote differentiation of functions with respect to their argument. For real argument and order,  $I$  and  $I'$  are real, but  $H$  and  $H'$  are complex; moreover, for the special values of the argument for which  $H$  is real it is always the case that  $H'$  is complex, and vice versa. Therefore equation (17) has no real solutions  $K$ , the surface wavenumber is a complex quantity, and we may write

$$K(\omega) = K_r(\omega) + iK_i(\omega). \tag{18}$$

In § 4 we shall solve equation (17) to obtain approximate expressions for  $K_r$  and  $K_i$ . First, however, we discuss the physical interpretation of a complex surface wavenumber. It will later emerge that  $K_i$  has the same sign as  $K_r$ . This means (equation (16)) that the surface wave decays as it rounds the bend (figure 2), as it must since radiation is appearing at infinity and none is being fed in. The attenuation is described by  $K_i$ . We expect, and shall later confirm, that  $K_r$  is very close to the value (12) found for  $K$  on a flat surface. Therefore  $K_r > k$  and the wave decays as  $z$  increases away from  $S$ . But at some

value of  $z$ ,  $z_r$  say, there must be a transition from decay to an outgoing complex oscillation which persists out to  $z = \infty$ . To find  $z_r$  we realize that the solution (16) corresponds to a wave with a definite constant angular momentum, proportional to  $K_r R$ . The linear transverse momentum, proportional to  $K_{it}$  (say) is *not* constant but decreases as  $r$  increases according to

$$K_{it}(r)r = K_r R \quad \text{ie } K_{it}(z) = \frac{K_r R}{R+z}. \quad (19)$$

Eventually  $K_{it}$  decreases to the vacuum wavenumber  $k$ , and there can be real propagating (non-evanescent) waves along  $z$ . This is the level  $z_r$  at which the transition to radiation occurs (figure 2), and equation (19) gives

$$z_r = \left( \frac{K_r}{k} - 1 \right) R. \quad (20)$$

This physical argument is confirmed by inspection of equation (16): the Bessel function  $H^{(1)}$  changes from decaying to oscillatory when its argument equals the real part of its order (Abramowitz and Stegun 1964, ch 9), and this occurs precisely at the level  $z = z_r$  given by equation (20).

Finally, we remark that the complex solutions  $K$  of equation (17) are special cases of the *Watson-Regge poles* of scattering theory (see eg Newton 1966). First we notice that  $K$  is proportional to the angular momentum number  $a$  of the basic solution (14). Next we realise that the purely outgoing waves we are seeking correspond to solutions of a *scattering problem* for which the amplitude ratio of scattered (outgoing) and incident waves is infinite. But this ratio is just the  $S$  matrix, so that the solutions of equation (17) also give the complex angular momentum numbers at which the poles of the  $S$  matrix are located, and these are precisely the *Watson-Regge poles*. However, this problem has no obvious precise quantum mechanical analogue, because of the unusual boundary condition on  $\nabla B$ .

#### 4. Cylindrical bends—asymptotic approximation for attenuation

Before outlining the solution of equation (17) we shall obtain formulae for  $K_r$  and  $K_i$  using physical arguments. The bends under consideration are gentle, so that  $kR \gg 1$ . Therefore there should be a surface wave closely similar to that on a flat surface (§ 2) and to a first approximation we expect the real part  $K_r$  of the surface wavenumber to be given by equation (12). To find the imaginary part  $K_i$  we use the conservation of energy, and equate the power lost travelling along  $S$  to the radiant power received at infinity. We assume, and our later results will confirm, that  $K_i \ll K_r$ , so that the surface wave is only slightly damped in one surface wavelength.

The wave decays along  $z$  for  $z < z_r$  (equation (20)); this range equals many surface wavelengths, so that the wave should closely resemble that on a flat surface over the range where it has significant magnitude (cf the sketches of  $B$  on figures 1 and 2). Therefore the total power  $P(\xi)$  flowing round the bend at  $\xi$  should be given by an argument virtually the same as that leading to equation (13). The only difference comes from the small decay exponent  $K_i \xi$  and leads to the result

$$P(\xi) = \frac{e^{-2K_i \xi} c^2 (|\epsilon|^2 - 1)}{4\mu_0 \omega |\epsilon|^{3/2}}. \quad (21)$$

Thus the power lost as the wave travels a small distance  $d\xi$  near  $\xi = 0$  is

$$-dP(0) = \frac{c^2(|\epsilon|^2 - 1)K_i d\xi}{2\mu_0\omega|\epsilon|^{3/2}}. \tag{22}$$

This power is radiated to infinity, the radiation emerging tangentially at  $z = z_r$  when the wave changes from evanescence to fully oscillatory. The energy will travel along straight lines (rays in free space), and elementary geometry (figure 3) shows that when

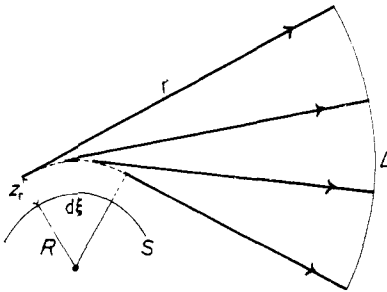


Figure 3. Power lost in  $d\xi$  emerges along  $L$ .

the power from  $d\xi$  reaches a large radius  $r$  it crosses normally a circular arc of length  $L$ , where

$$L = \frac{r d\xi}{R}. \tag{23}$$

Therefore

$$-dP(0) = \Pi_r(\xi = 0, r \rightarrow \infty)L \tag{24}$$

where  $\Pi_r$  is the magnitude of the Poynting vector.

Equations (22), (24), (10) and (16) then give

$$K_i = \frac{k|\epsilon|^{3/2}}{(|\epsilon|^2 - 1)R|H_{K_r,R}^{(1)}(kR)|^2} \lim_{r \rightarrow \infty} r \operatorname{Im} H_{K_r,R}^{(1)*}(kr)H_{K_r,R}^{(1)'}(kr) \tag{25}$$

where we have written  $K_r$  for  $K$  since we are evaluating  $K_i$  to first order only. For the Bessel functions we employ standard asymptotic forms. In the numerator the arguments of  $H^*$  and  $H'$  are  $kr$ , so that we have ‘argument  $\gg$  order’ and formula (9.2.3) of Abramowitz and Stegun (1964) leads to

$$\lim_{r \rightarrow \infty} r \operatorname{Im} H_{K_r,R}^{(1)*}(kr)H_{K_r,R}^{(1)'}(kr) = \frac{2}{\pi k}. \tag{26}$$

In the denominator both the argument  $kR$  and order  $K_r R$  are large (gentle bend) but  $K_r > k$  (equation (12)), so that we have ‘argument  $<$  order, order large’, which calls for the ‘Debye formulae’ (Abramowitz and Stegun 1964, p 366). Neglecting exponentially small terms we may approximate  $H^{(1)}$  by  $iY$ , where  $Y$  is the Bessel function of the second kind, and we obtain

$$|H_{K_r,R}^{(1)}(kR)|^2 \approx \frac{2 \exp(2K_r R \{ \ln[K_r/k + (1 + K_r^2/k^2)^{1/2}] - (1 - k^2/K_r^2)^{1/2} \})}{\pi K_r R (1 - k^2/K_r^2)} \tag{27}$$



Substituting equations (26) and (27) into (25) and using equation (12) for  $K_r$ , we obtain, after a little algebra :

$$K_r = k \left( \frac{|\epsilon|}{|\epsilon| - 1} \right)^{1/2}$$

$$K_i = \frac{k|\epsilon|^{3/2}}{(|\epsilon| + 1)(|\epsilon| - 1)^{3/2}} \exp \left\{ -\frac{kR}{(|\epsilon| - 1)^{1/2}} \left[ \sqrt{|\epsilon|} \ln \left( \frac{\sqrt{|\epsilon| + 1}}{\sqrt{|\epsilon| - 1}} \right) - 2 \right] \right\} \quad (28)$$

These are the main results of this section. They can also be obtained directly from equation (17), in a less physically instructive way, as follows: once again we have 'argument < order, order large' and we can employ the Debye formulae for the Bessel functions. Now we decompose  $H_{KR}^{(1)}$  into its real and imaginary parts, that is

$$H_{KR}^{(1)} = J_{KR} + iY_{KR}. \quad (29)$$

As before, the  $iY$  term dominates exponentially over the  $J$  term. However, if we omit the  $J$  term and use the asymptotic forms we find that equation (17) is satisfied by  $K = K_r$ , where  $K_r$  is given by the first part of equation (28), *without any imaginary part*  $K_i$ . The reason for this is that replacing  $H$  by  $iY$  implies the use of standing wave solutions instead of the outgoing solutions (16), and this corresponds to a non-decaying surface wave being fed by an incident wave at the same rate as it radiates. If we do include the  $J$  term in equation (29), and treat it as a perturbation, we obtain a small imaginary correction  $iK_i$  to  $K_r$ , where  $K_i$  is given exactly by the second part of equation (28).

We now study the 'attenuation ratio'  $q$  of the imaginary and real parts of  $K$ . Equations (28) give

$$q = \frac{|\epsilon|}{|\epsilon|^2 - 1} \exp \left\{ -\frac{kR}{(|\epsilon| - 1)^{1/2}} \left[ \sqrt{|\epsilon|} \ln \left( \frac{\sqrt{|\epsilon| + 1}}{\sqrt{|\epsilon| - 1}} \right) - 2 \right] \right\}. \quad (30)$$

In terms of  $q$  the power flow (21) decays in one surface wavelength  $2\pi K_r$ , by the factor

$$F_1 = \frac{P[\xi + (2\pi/K_r)]}{P(\xi)} = e^{-4\pi q} \quad (31)$$

while in travelling one radian round the bend the decay factor is

$$F_2 \equiv \frac{P(\xi + R)}{P(\xi)} = \exp[-2kRq(|\epsilon|/|\epsilon| - 1)^{1/2}]. \quad (32)$$

Our analysis is valid only if  $q \ll 1$ , and equation (30) yields, encouragingly, the following limiting cases:

$$q = 0 \quad \text{if } kR \rightarrow \infty \text{ (flat surface)}$$

or

$$|\epsilon| \rightarrow 1 \text{ (threshold at which surface waves appear—corresponding to } \omega = \omega_p/\sqrt{2} \text{ on free electron model)}$$

or

$$|\epsilon| \rightarrow \infty \text{ (ie } \omega \rightarrow 0 \text{ on free electron model).} \quad (33)$$

The last case,  $|\epsilon| \rightarrow \infty$  calls for comment. By using the Debye approximations for the Hankel functions  $H_{KR}^{(1)}(kR)$ , we have assumed that the argument is less than the order.

But as  $|\epsilon| \rightarrow \infty$ ,  $K_r \rightarrow k$  (equation (28)) so that the argument becomes *equal* to the order and our formulae cease to be valid. Analysis of the Debye approximations using the ‘transitional’ formulae (especially (9.3.24) of Abramowitz and Stegun 1964) shows that the following condition will guarantee the applicability of the basic results (28) and (30):

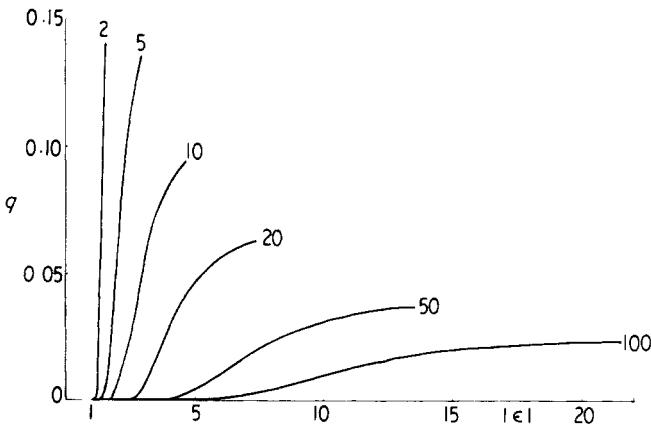
$$|\epsilon| \ll (\frac{1}{2}kR)^{2/3} \quad \text{ie } \omega > \omega_p \left( \frac{2}{kR} \right)^{1/3} \quad \text{on free electron model.} \quad (34)$$

In the low-frequency limit when this condition is violated, the argument and order of  $H^{(1)}$  are so nearly equal that the wave hardly decays away from the surface before the height  $z_r$  (equation (20)) is reached and radiation begins—in these circumstances a ‘bound surface wave’ can hardly be said to exist. It might be objected that this analysis of the regime  $|\epsilon| \rightarrow \infty$  is unrealistic, because as  $\omega \rightarrow 0$  dissipative effects become important (equation (6)) and we leave the frequency range (equation (7)) in which  $\epsilon$  is real, so that an undamped surface wave cannot exist, even on a flat surface. However, dissipative effects are unimportant unless  $\omega \lesssim \tau^{-1}$  and  $\tau^{-1}$  is less than the limiting frequency in equation (34) provided

$$kR < 2(\omega_p \tau)^3 \sim 10^{21} \quad (34a)$$

typically, and this condition will in practice always be satisfied. Therefore, there is always a frequency regime in which dissipative effects are unimportant but where rapid radiative attenuation of surface waves occurs and formulae (28) and (30) are invalid (the Watson–Regge poles  $K$  will lie far from the real axis). This emphasizes the fact that the radiative attenuation that is the subject of this paper is fundamentally different from, and not to be confused with, attenuation due to resistive dissipation associated with an imaginary part of  $\epsilon$ .

From the limits (33) it is clear that as  $|\epsilon|$  increases from 1 to  $\infty$ , with  $kR$  held fixed,  $q$  as given by equation (30) increases at first and then decreases. However, the maximum value occurs when  $|\epsilon| \sim (kR)^{2/3}$ , and equation (34) shows that this is outside the range of validity of equation (30). Therefore  $q$  is an increasing function of  $|\epsilon|$  over the range of validity of equations (28) and (30). Figure 4 shows curves of  $q(|\epsilon|)$  for  $1 < |\epsilon| < (kR)^{2/3}$  computed for various values of  $kR$ .



**Figure 4.** Attenuation ratio  $q$  (equation (30)) as a function of  $|\epsilon|$  in the range  $1 < |\epsilon| < (kR)^{2/3}$  for the fixed values of  $kR$  marked on the curves.

Finally, we point out that Lapin (1969) has performed a similar analysis for *acoustic* surface waves on a solid cylinder surrounded by liquid.

### 5. Gently curved surfaces of general form

No rigorous results exist for general curved surfaces so this section will be somewhat speculative. We begin with surfaces  $S$  curved in the  $xz$  plane but not along  $y$ , and specified by their varying radius of curvature  $R(\xi)$ , where  $\xi$  is the distance along  $S$  measured from an arbitrary origin; we take  $R$  positive if  $S$  is convex towards the vacuum.

If  $S$  is gently curved,  $R$  varies slowly, and if  $R$  is positive we can treat  $S$  as locally convex cylindrical and apply the theory of the previous two sections. Then the local attenuation at  $\xi$  is governed by  $K_i(\xi, \omega)$ , obtained by setting  $R = R(\xi)$  in the second of equations (28). An obvious generalization of the arguments leading to equations (21) and (22) gives the power flowing past the point  $\xi$  in terms of the power flowing past  $\xi = 0$  as

$$P(\xi) = P(0) \exp\left(-2 \int_0^\xi d\xi' K_i(\xi', \omega)\right) \tag{35}$$

provided  $\xi$  increases in the direction of propagation of the surface wave.

To deal with parts of  $S$  that are *concave towards the vacuum*, for which  $R(\xi) < 0$  we must repeat the analysis of §3 for cylinders filled with vacuum and surrounded by dielectric. The surprising result emerges that it is possible in this case to have a true surface wave which is nowhere oscillatory or radiative, whose form is given not by equation (16) but by

$$\begin{aligned} B(\xi, z) &= e^{iK\xi} K_{K|R|}(k\sqrt{|\epsilon|(|R| + z)}) [K_{K|R|}(k\sqrt{|\epsilon||R|})]^{-1} & (z > 0) \\ &= e^{iK\xi} J_{K|R|}(k(|R| + z)) [J_{K|R|}(k|R|)]^{-1} & (z < 0) \end{aligned} \tag{36}$$

where  $K_{K|R|}$  is the modified Bessel function of the second kind. This leads to an equation for the surface wavenumber that is analogous to equation (17) but is purely real with a real solution that for large  $k|R|$  becomes precisely  $K_r$  as given by the first of equations (28). This strongly suggests that *surface waves on concave parts of  $S$  are not attenuated*, a conjecture supported by the observation that as we approach a point of inflexion on  $S$  along a convex part of  $S$ , the attenuation  $K_i(\xi)$ , as given by equation (28), vanishes exponentially.

Up to now we have considered  $S$  as locally cylindrical. Now we ask: what will be the effects of *locally changing curvature*? Recall first that the introduction of a locally changing slope, ie curvature, did not alter the real part  $K_r$  of the surface wavenumber but introduced a qualitatively different effect, namely radiative attenuation described by  $K_i$ . Similarly, we expect that the introduction of a locally changing curvature will not alter  $K_r$  or  $K_i$ , but will introduce another qualitatively different effect; we conjecture that this is *reflection of the surface wave*. It seems very difficult to estimate the magnitude of this effect, but arguing by analogy with the quantum mechanical reflection of waves above a smooth potential barrier (Berry and Mount 1972) we suggest that the reflection will be exponentially small—more precisely of order

$$\exp\{-k^2/[d(\text{curvature})/d\xi]\} = \exp(-k^2 R^2/R') \tag{37}$$

—for gently curving ‘analytic’ surfaces none of whose derivatives  $R'(\xi)$ ,  $R''(\xi)$  etc is

discontinuous. If equation (37) is correct, reflection will be an even weaker effect than attenuation, which from equation (28) is of order  $e^{-kR}$ .

It is not possible to test these conjectures by solving the wave equation (4) exactly for some idealized surface  $S$  with changing curvature. The simplest case would seem to have  $R'(\xi)$  constant, but this is a spiral, which is hardly promising. Alternatively, we might choose  $S$  to be a coordinate surface of the elliptic or parabolic cylindrical system, for which the operator  $\nabla^2$  separates. However, we quickly discover that equation (4) is not separable in these systems (see Morse and Feshbach 1953, p 513); the point—which does not seem always to be appreciated (II, p 396)—is that the wave equation is separable if any of these coordinate surfaces  $S$  is a perfect reflector on which the solution or its normal derivative vanishes, but is not separable if  $S$  bounds two regions in which waves can exist.

Finally, we consider waves on surfaces gently bending in  $x$ ,  $y$  and  $z$ , that is surfaces with two principal curvatures  $R_1^{-1}$  and  $R_2^{-1}$  at each point, restricted only in that  $kR_1$  and  $kR_2$  must both be large. Our previous curved surfaces  $S$  were locally cylindrical, so that one curvature vanished; the *Gaussian curvature*  $(R_1 R_2)^{-1}$  was zero, and  $S$  was 'developable' from a plane without stretching or tearing. When the Gaussian curvature is non-zero,  $S$  is still locally flat, and surface waves still exist, with surface wavenumbers whose real part  $K_r$  is given by the first of equations (28). This is proved in the appendix. Because of the short wavelengths, the waves will, to a close approximation, propagate according to the laws of geometrical optics (see appendix), that is their energy will travel along *rays*, confined to the surface and satisfying Fermat's principle of least time (Keller 1958, Lewis *et al* 1967). If  $S$  is time-independent and the underlying dielectric spatially homogeneous, this implies that *the rays are geodesics on S*.

If the surface rays are considered as space curves  $\mathbf{r}(\xi) = (x(\xi), y(\xi), z(\xi))$ , where  $\xi$  is arc length, then they are completely determined by specifying the initial conditions  $\mathbf{r}(0)$  (which must lie on  $S$ ) and  $\mathbf{r}'(0)$  (which must lie in the tangent plane to  $S$  at  $\mathbf{r}(0)$ ). Standard differential geometry (eg Stavroudis 1972) then gives the result that at any point on the ray the principal normal  $\mathbf{n}(\xi)$ , defined as

$$\mathbf{n}(\xi) \equiv \frac{\mathbf{r}''(\xi)}{|\mathbf{r}''(\xi)|} \tag{38}$$

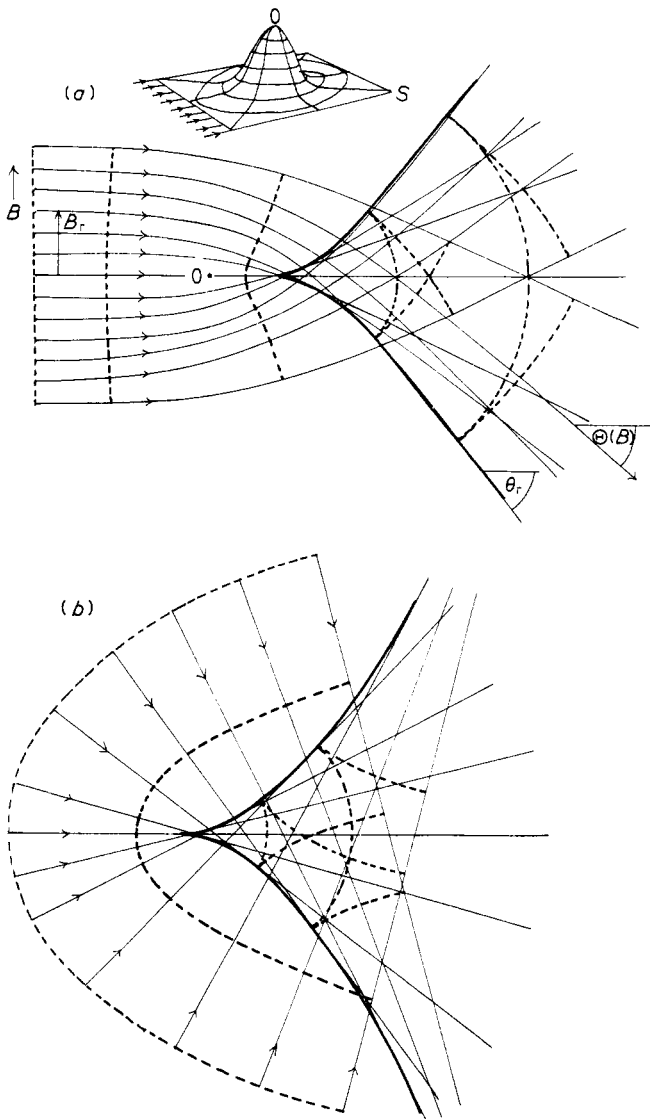
lies along the normal to  $S$  at  $\mathbf{r}(\xi)$ . This 'dynamical equation' (which is analogous to Newton's second law for particle trajectories or Snell's law for the refraction of rays in space) can be used to derive the *curvature*  $\mathcal{K}(\xi)$  and *torsion*  $\mathcal{T}(\xi)$  of the surface rays at  $\xi$ . If the ray tangent  $\mathbf{r}'(\xi)$  makes an angle  $\theta$  with the principal direction on  $S$  corresponding to the curvature  $R_1^{-1}$ , then  $\mathcal{K}$  and  $\mathcal{T}$  are given by

$$\begin{aligned} \mathcal{K}(\xi) &\equiv |\mathbf{r}''(\xi)| = \left| \frac{\cos^2\theta}{R_1} + \frac{\sin^2\theta}{R_2} \right| \\ \mathcal{T}(\xi) &\equiv \frac{(\mathbf{r}'(\xi) \times \mathbf{r}''(\xi)) \cdot \mathbf{r}'''(\xi)}{|\mathbf{r}''(\xi)|^2} = \frac{1}{2} \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \sin 2\theta. \end{aligned} \tag{39}$$

The cases previously considered correspond to taking  $R_2 = \infty$  and  $\theta = 0$ ; then the ray is a plane curve (torsion-free) on a locally cylindrical surface. Ray propagation on surfaces that can be referred to a reference plane will be considered in detail in § 6.

A single ray does not define a surface wave: it is necessary to have a *family* of rays, whose orthogonal lines are the surface wavefronts (see appendix). This raises the new possibility of *focusing* of surface waves where an initially smooth wavefront develops

kinks; these kinks lie on *caustic lines*, that is on envelopes of the ray family. The caustic lines themselves are smooth except for isolated singular points at which the caustic has a *cusp*; this is a consequence of the topological 'theory of catastrophes' (Thom 1969, 1975). Cusped caustics can arise on curved surfaces from a straight initial wavefront (figure 5(a)), or on flat surfaces from an initial wavefront that is a noncircular arc concave towards the propagation direction (figure 5(b)). In the next section we shall show how caustics are generated when surface rays are refracted by a hill on  $S$ . In the present context the importance of caustics lies in the powerful intensification of the surface wave in their neighbourhood. According to geometrical optics (see appendix) the intensity  $|B|^2$  would be infinite at a caustic on  $S$ , but it is precisely on caustics that geometrical



**Figure 5.** Patterns of surface ray tracks (—), wavefronts (---) and caustics (——) (a) on a curved surface  $S$  and (b) on a flat surface (straight tracks).

optics breaks down, and it can be shown (eg Pearcey 1946, § 4 of Berry 1975) that the intensity rises to large values given not by formulae (A.1) and (A.9) but by

$$|B|^2 = \begin{cases} O[(K,R)^{1/2}] & \text{on a cusp of a caustic on } S \\ O[(K,R)^{1/3}] & \text{on a smooth part of a caustic on } S \\ O[1] & \text{on a point on } S \text{ that does not lie on} \\ & \text{a caustic.} \end{cases} \quad (40)$$

Waves traversing general curved surfaces should of course radiate whenever  $S$  is convex to the vacuum, by a mechanism essentially the same as that described in §§3 and 4. The attenuation should be given by equations (28) or (30), where instead of  $R$  it is necessary to use the radius of curvature  $\mathcal{K}^{-1}$  (equation (39)) of the ray at the point considered. (This is borne out by analysis of surface waves travelling *parallel to the axis* of a cylinder: the surface curvature is zero and there is no attenuation whether the dielectric is convex or concave to the vacuum.) The radiation will appear along a line parallel to the tangent to the surface ray at the point concerned, lying at a perpendicular distance  $z$ , above the surface (equation (20)). In addition, as discussed earlier, we would expect very weak reflection at places where the surface ray curvature is changing.

**6. Cusped rainbows in waves focused by hills**

In order to justify the remarks made in § 5 about caustics (figure 5(a)) of families of surface rays, we consider in detail geodesics on a surface  $S$  whose height  $z$  above a reference plane  $R = (x, y)$  is (figure 6)

$$z = f(R). \quad (41)$$

We call the direction of increasing  $z$  the ‘upward’ direction. For a ray (geodesic) the arc length  $\xi$  on  $S$  is an extremum. We describe the rays by their projections  $R(l)$  on the plane  $R$ , where  $l$  is arc length on  $R$  (figure 6). We call  $R(l)$  the *track* of the ray. Then, for

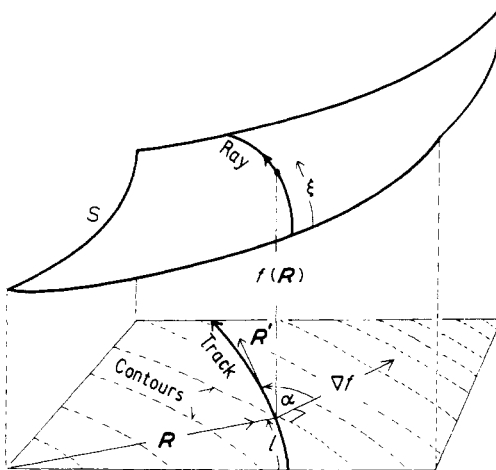


Figure 6. Ray on  $S$  and track on  $R$  plane.

any path between two points on  $S$  we have

$$\xi = \int d\xi = \int dl[R'^2 + (\mathbf{R}' \cdot \nabla f)^2]^{1/2} \quad (42)$$

where  $\mathbf{R}' \equiv d\mathbf{R}/dl$  is the unit tangent vector of the corresponding track. The 'Lagrangian' function is

$$L(\mathbf{R}, \mathbf{R}') = [R'^2 + (\mathbf{R}' \cdot \nabla f)^2]^{1/2} \quad (43)$$

From this can be derived the Euler-Lagrange equations for the tracks, namely

$$\mathbf{R}'' = (\mathbf{R}'(\mathbf{R}' \cdot \nabla f) - \nabla f)(\mathbf{R}' \cdot \nabla)^2 f \quad (44)$$

where we have written a form valid when  $|\nabla f|^4$  can be neglected (in practice this will be a good approximation if the slope of  $S$  relative to the  $\mathbf{R}$  plane never exceeds  $30^\circ$ ). It is amusing to note that the same equation can be derived from the following 'Hamiltonian' function involving the canonical momentum  $\mathbf{P}$  conjugate to  $\mathbf{R}$ :

$$H(\mathbf{R}, \mathbf{P}) = \frac{1}{2}(P^2 - (\mathbf{P} \cdot \nabla f)^2/1 + (\nabla f)^2). \quad (45)$$

(This gives an equation for  $\mathbf{R}$  as a function of  $\xi$ , and it is necessary to change parameters from  $\xi$  to  $l$  in order to obtain equation (44).) The equations of geodesics can also be obtained from the Hamilton-Jacobi equation (A.7) for the wavefronts (Lewis *et al* 1967).

The 'dynamical equation' (44) enables the curvature  $\mathcal{C}$  of the track to be calculated.  $\mathcal{C}$  is defined as

$$\mathcal{C} \equiv (\mathbf{R}' \times \mathbf{R}'') \cdot \hat{\mathbf{z}} \quad (46)$$

and must be distinguished from the spatial curvature  $\mathcal{K}$  (equation (39)) of the ray. The track is turning to the left if  $\mathcal{C} > 0$ , and to the right if  $\mathcal{C} < 0$ , as seen by someone looking along  $\mathbf{R}'$  with his head pointing along  $+\hat{\mathbf{z}}$ . Let  $\alpha$  be the angle measured clockwise from the vector  $\nabla f$  to the track tangent  $\mathbf{R}'$  (figure 6), and let  $C_{\mathbf{R}'}$  be the upward curvature of the ray (not the track), ie

$$C_{\mathbf{R}'} \equiv (\mathbf{R}' \cdot \nabla)^2 f(\mathbf{R}). \quad (47)$$

Then from equation (46) we obtain

$$\mathcal{C} = |\nabla f| \sin \alpha C_{\mathbf{R}'}. \quad (48)$$

Now if  $S$  is represented as a map on  $\mathbf{R}$  by contours  $f(\mathbf{R}) = \text{constant}$ ,  $\nabla f$  points along the lines of steepest ascent on the map, ie perpendicular to the contours. Equation (48) is analogous to Snell's law of refraction for rays in smoothly inhomogeneous media: its qualitative content may be expressed thus: if the ray curves upwards/downwards, the track turns away from/towards lines of steepest ascent on the  $\mathbf{R}$ -map of  $S$ . The implications of this rule are different according to whether the Gaussian curvature of  $S$  (that is the product of principal curvatures  $R_1^{-1}$  and  $R_2^{-1}$ ) is positive or negative in the region considered. If the Gaussian curvature is positive then  $S$  is wholly concave or convex, and all rays, whatever their direction, curve upwards ( $S$  concave) or downwards ( $S$  convex); then the refraction law can be stated in the following 'invariant' form which is independent both of the sense in which the track is traversed and of the sense of  $\hat{\mathbf{z}}$ : the normal to the track at any point  $\mathbf{R}$  makes an angle not exceeding  $90^\circ$  with the normal to the contour passing through  $\mathbf{R}$  provided both normals point away from the convex

sides of their curves. If the Gaussian curvature is negative (so that  $S$  is locally saddle-shaped) then some rays curve upwards and some downwards, depending on their direction, and the corresponding invariant refraction law is more complicated.

Now we specialize to surfaces  $S$  with rotational symmetry about the  $z$  axis, so that in terms of plane polar coordinates  $\mathbf{R} = (R, \theta)$  the height function  $f(\mathbf{R})$  depends on  $R$  only. We call such surfaces ‘hills’, this terminology emphasizing that we shall consider only cases where  $f(R \rightarrow \infty) = 0$ . (Of course geodesics are independent of the sign of  $f$ , so that every hill has an equivalent depression.) Geodesics on a hill will conserve their ‘angular momentum’  $B$ . This quantity is defined in terms of the Lagrangian (43) as

$$B \equiv \frac{\partial L}{\partial \theta'} = \frac{R^2 \theta'}{[R'^2 + R^2 \theta'^2 + R'^2 (df/dR)^2]^{1/2}} \tag{49}$$

from which we derive the polar equation for the track  $R(\theta)$ , namely

$$\frac{d\theta}{dR} = \pm \frac{B[1 + (df/dR)^2]^{1/2}}{R(R^2 - B^2)^{1/2}}; \tag{50}$$

it is easy to show that  $B$  is the *impact parameter* of the geodesic, that is its distance at  $R = \infty$  from the parallel track through  $R = 0$  (figure 5(a)).

We have here a situation closely analogous to the classical scattering of particles by a spherically symmetrical force field—‘potential scattering’—and it is natural (Ford and Wheeler 1959a, b, Berry and Mount 1972) to calculate the *deflection function*  $\Theta(B)$  which gives the total deflection (figure 5(a)) of a track along its whole infinite length. From equation (50) we obtain

$$\Theta(B) = \pi - 2B \int_B^\infty \frac{dR[1 + (df/dR)^2]^{1/2}}{R(R^2 - B^2)^{1/2}} \tag{51}$$

This is always negative for  $B > 0$ , which means that *all hills are attractive*. This contrasts with the case of potential scattering, as does the fact, obvious from equation (50), that the distance of closest approach is simply  $R = B$ , whatever the shape  $f(R)$  of the hill. If the hills are small perturbations of the plane ( $|df/dR| \ll 1$ ), then equation (50) takes the simpler form

$$\Theta(B) = -B \int_B^\infty \frac{dR(df/dR)^2}{R(R^2 - B^2)^{1/2}} \tag{52}$$

closely analogous to the deflection function for particle scattering at high energies, with  $(df/dR)^2$  playing the role of the potential. We can obtain an explicit expression for  $\Theta(B)$  if the hill is Gaussian, ie

$$f(R) = f_0 \exp(-R^2/2a^2). \tag{53}$$

From equation (52) the expression is

$$\Theta(B) = -\frac{\sqrt{\pi} f_0^2 B}{a^3} \exp(-B^2/a^2). \tag{54}$$

Next we study the focusing by a hill of a family of ray tracks that are initially parallel and described by the values of  $B$  from  $-\infty$  to  $+\infty$  (figure 5(a)); this is analogous to the scattering of a beam of particles by a potential. At infinity there is focusing at the *rainbow angle*  $\theta_r$  (Ford and Wheeler 1959a, b, Berry 1966a), defined (figure 5(a)) by

$$\theta_r = -\Theta(B_r) \quad \text{where } d\Theta(B_r)/dB = 0. \tag{55}$$



For the Gaussian hill (53) we have

$$\theta_r = \sqrt{\left(\frac{\pi}{2e}\right) \frac{f_0^2}{a^2}} = 0.7602 \frac{f_0^2}{a^2}. \tag{56}$$

In the near region there will be a paraxial focus at  $R = R_c$ , defined as the radius at which rays with vanishingly small impact parameters  $B$  cross the forward direction  $\theta = 0$ . If  $|(df/dR)| \ll 1$ ,  $R_c$  is given by

$$R_c = \lim_{B \rightarrow \infty} \frac{B}{|\Theta(B)|} = \frac{1}{|d\Theta(0)/dB|} = \frac{1}{\int_0^\infty (dR/R^2)(df/dR)^2} \tag{57}$$

which for a Gaussian hill is

$$R_c = \frac{a^3}{f_0^2 \sqrt{\pi}}. \tag{58}$$

The point  $(R_c, 0)$  is the cusp of a caustic joining the focus to the two rainbows at infinity. This caustic  $\theta_c(R)$  is the envelope of the family of ray tracks (figure 5(a)), defined as follows: let  $\theta(R, B)$  be the outgoing part of the track, obtained from equation (50), of the ray with impact parameter  $B$  (so that  $\theta(\infty, B) = \Theta(B)$ ). Then two rays touch at  $R$  if

$$B = B_c(R) \quad \text{where} \quad \frac{\partial \theta(R, B_c)}{\partial B} = 0 \tag{59}$$

and the caustic is

$$\theta_c(R) = \theta(R, B_c(R)). \tag{60}$$

Some algebra shows that near the focus  $R_c$  the caustic is given explicitly by

$$\theta_c(R) = \pm \frac{2\sqrt{2}[(R/R_c) - 1]^{3/2}}{3[1 + R_c^3(d^3\Theta(0)/dB^3)]^{1/2}} \tag{61}$$

which does indeed have the necessary cusped form.

Surprisingly, it is possible to achieve *perfect focusing* at any chosen radius  $R = R_c$  with a finite bundle of rays whose impact parameters  $B$  lie between  $\pm R_m$ , by suitably choosing the shape of the hill. Such a 'perfect geodesic lens' has finite radius  $R_m$  and infinite slope at  $R = R_m$ . The lens shape  $f(R)$  can be found from the ray equation (50) by requiring all rays striking the lens to pass through the point  $R = R_c$ ,  $\theta = 0$  and transforming this condition into Abel's integral equation (Bôcher 1926). The result is that  $f(R)$  is given by

$$\{1 + [df(R)/dR]^2\}^{1/2} - 1 = \frac{1}{\pi} \left[ \frac{R_m}{(R_m^2 - R^2)^{1/2}} \sin^{-1} \frac{R_m}{R_c} - \sin^{-1} \left( \frac{R_m^2 - R^2}{R_c^2 - R^2} \right)^{1/2} \right] \tag{62}$$

We emphasize that this result is not restricted to paraxial rays. It is even possible to make rays focus on the far edge of the lens by taking  $R_c = R_m$ ; then equation (62) gives  $f(R)$  as

$$f(R) = \frac{R_m}{2} \left( \cos^{-1} \frac{R}{R_m} + \frac{R}{R_m} - 1 \right). \tag{63}$$

This function and the associated ray tracks are plotted in figure 7. These 'lenses' are completely aberration-free: obviously there is no chromatic aberration (the geodesic rainbows (55) and (56) are not coloured), and there is no 'circular aberration' either, because the symmetry means that all bundles of parallel rays from whatever direction will focus on the circle  $R = R_c$ .

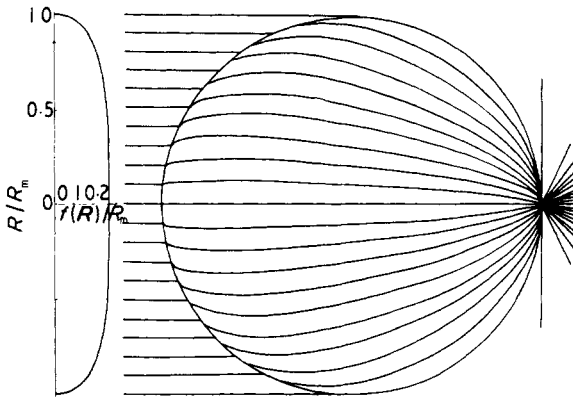


Figure 7. Shape and ray tracks of the perfect geodesic lens for  $R_c = R_m$

At the edge of the geodesic lens the surface  $S$  is certainly not 'gently bent' and in practice the focus would be very weak because most of the rays would be reflected instead of being refracted onto the hill. This could be avoided by smoothing the discontinuity in slope at  $R = R_c$  so that all radii of curvature exceed the surface wavelength  $2\pi/K$ .

Finally, we remark that discontinuities in the slope of  $S$  cause ray tracks to change direction in accordance with an analogue of Snell's law for discontinuous media. This can be derived from the Lagrangian (43), since the component of canonical momentum parallel to the discontinuity is conserved; the refraction law for the case where the discontinuity is horizontal is

$$\frac{\sin \theta}{[1 + \cos^2 \theta (\nabla f_n)^2]^{1/2}} = \text{constant across the discontinuity} \quad (64)$$

where  $\theta$  is the angle made by the ray track with the normal to the discontinuity, and  $(\nabla f_n)$  is the component of the slope of  $S$  normal to the discontinuity.

A discussion of focusing of electromagnetic surface waves has been given by Shubert and Harris (1971), while Van Duzer (1970) and Mason (1973) consider the analogous problem for acoustic surface waves. These authors do not discuss caustics and rainbows, or aberration-free lenses, but they do consider lenses of different material from the bulk, and 'surface wave guides' in the form of grooves. Recently Bell *et al* (1975) have claimed to study refraction of surface waves by dielectric prisms and cylinders on the vacuum side of a flat aluminium surface. However, for the microwave frequencies they used, equation (12) gives  $K_r \sim k$ , so that the decay into the vacuum is extremely slow and the authors are really studying not surface waves but ordinary plane waves in the half-space above the aluminium.

## 7. Discussion

The foregoing theory suggests a number of experiments. If a surface wave radiates tangentially (figure 2) when rounding a bend, then it must be possible (from the 'incoming' solution analogous to equation (16)) to excite a surface wave by tangentially *illuminating* a bend. Both predictions could be tested with the surface geometry sketched in figure 8.

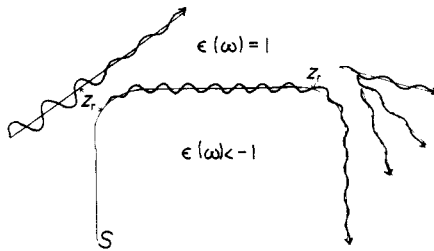


Figure 8. Proposed experiment to test attenuation.

The radiation should appear to come not from the surface of the bend itself but from the height  $z_r$  given by equation (20). By varying the frequency and the radius of curvature of the bends, the basic formulae (28)–(32) could be tested. (The distance between the bends should of course be much less than the 'dissipation length' associated with the relaxation time  $\tau$ .)

Any of the various focal structures discussed in § 6 could be produced by forming  $S$  into a hill of the appropriate shape. If the focus is made to occur at a point where the geodesics have a large spatial curvature convex towards the vacuum, then the focus should be a source of intense tangential radiation away from  $S$ . One way to achieve this would be to place a hill between the two bends in figure 8, with the focus falling on the second bend. The same geometry could be employed in reverse to excite a plane wave in a surface by intense tangential illumination of a point.

Concerning the theory, it would be desirable to have a more rigorous treatment of the questions discussed in § 5. Is there really no attenuation of a wave rounding a concave bend, or is the attenuation just very small? Exactly how much of a wave is reflected by a smooth change of curvature of  $S$ ? Is it true that for a general ray, bending and twisting on a doubly-curved surface, the attenuation depends only on the local spatial curvature of the ray?

## Acknowledgments

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## Appendix

We show here that 'p polarized' surface waves will exist on any gently bent surface  $S$ . To label a general space point  $P$  we use its perpendicular distance  $z$  above  $S$  (ie  $z < 0$ )

in the dielectric below  $S$ ) and two coordinates  $(\xi, \eta)$  giving the position on  $S$  of the foot of the perpendicular from  $P$ . This labelling will be unique if  $P$  is sufficiently close to  $S$  for no centre of curvature of  $S$  to lie on the perpendicular between  $P$  and  $S$ . If a surface wave exists, its form will be the following generalization of equations (8) and (11):

$$\mathbf{B}(z, \xi, \eta) = \mathbf{u}(\xi, \eta) \exp[ik\phi(\xi, \eta)] \times \begin{cases} \exp(-kq_>z) & (z > 0) \\ \exp(+kq_<z) & (z < 0). \end{cases} \tag{A.1}$$

Lines of constant phase  $\phi$  are the surface wavefronts, and the normals are surface rays, the surface wavevector being

$$\mathbf{K}(\xi, \eta) = k\nabla_S\phi(\xi, \eta) \tag{A.2}$$

where  $\nabla_S$  denotes gradient parallel to  $S$ . The magnitude of  $\mathbf{u}$  gives the intensity of the wave and will describe focusing, while the direction of  $\mathbf{u}$  gives the polarization. For  $p$  polarization we require  $\mathbf{u} \cdot \hat{\mathbf{z}}$  to vanish. The decay exponents  $q_>$  and  $q_<$  are related by the boundary condition on  $\mathbf{B}$ , and the appropriate generalization of equation (9), namely that  $(\nabla \times \mathbf{B}) \times \hat{\mathbf{z}}/\epsilon$  is continuous across  $S$ , gives

$$q_< = -\epsilon q_> \quad \text{ie } q_< = |\epsilon|q_>. \tag{A.3}$$

Of course equation (A.1) is not an exact solution of the wave equation (4), but we do expect it to be a ‘geometrical optics’ approximation valid in the asymptotic limit  $k \rightarrow \infty$  corresponding to gentle bending of  $S$  on a wavelength scale. To illustrate this we demand that equation (A.1) cause  $\nabla \cdot \mathbf{B}$  to vanish, and we obtain

$$\nabla_S \cdot \mathbf{u} + ik\mathbf{u} \cdot \nabla_S\phi = 0. \tag{A.4}$$

For large  $k$  the second term dominates and then equation (A.2) tells us that  $\mathbf{u} \cdot \mathbf{K}$  vanishes —ie the wave is transverse, as expected.

Now we substitute equation (A.1) into (4); this gives

$$k^2 \left[ -(\nabla_S\phi)^2 + \begin{cases} q_>^2 + 1 & \text{if } z > 0 \\ q_<^2 + \epsilon & \text{if } z < 0 \end{cases} \right] \mathbf{u} + ik[\mathbf{u}\nabla_S^2\phi + 2\nabla_S\phi \cdot \nabla\mathbf{u}] + \nabla_S^2\mathbf{u} = 0. \tag{A.5}$$

The term in  $k^2$  dominates; equating its coefficient to zero gives an equation satisfied by the phase  $\phi$ . Now  $\phi$  must be independent of  $z$ , and this implies that

$$q_>^2 + 1 = q_<^2 + \epsilon. \tag{A.6}$$

Combining this with (A.3), we get the following ‘Hamilton–Jacobi’ equation for  $\phi$ :

$$(\nabla_S\phi)^2 = \frac{\epsilon}{\epsilon + 1} = \frac{|\epsilon|}{|\epsilon| - 1} \tag{A.7}$$

Moreover, this gives with equation (A.2) a surface wavenumber  $|\mathbf{K}|$  exactly the same as the value (12) for a flat surface.

If we set equal to zero the term of order  $k$  in (A.5), we obtain after a little manipulation the following equation for the intensity  $|\mathbf{u}|^2$ :

$$\mathbf{K} \cdot \nabla_S \ln(|\mathbf{u}|^{-2}) = \nabla_S \cdot \mathbf{K}. \tag{A.8}$$

Now  $\mathbf{K} \cdot \nabla_S$  measures the rate of change along a ray, and  $\nabla_S \cdot \mathbf{K}$  measures the divergence of ray bundles, and if  $\xi$  is arc length on  $S$  along a ray and  $w(\xi)$  is the width of a narrow

ray bundle at  $\xi$  then equation (A.8) can be solved (see eg Berry 1966b, p 21) to give

$$|\mathbf{u}(\xi)| = |\mathbf{u}(0)| \left( \frac{w(0)}{w(\xi)} \right)^{1/2} \quad (\text{A.9})$$

This is obviously the correct focusing law for families of surface rays. Points for which  $w(\xi)$  is zero lie on caustics as discussed in §§ 5 and 6.

We conclude that on general gently bent surfaces  $p$  polarized waves do indeed exist. On surfaces that are not gently bent the term  $\nabla_{\xi}^2 \mathbf{u}$  in equation (A.5) will become important and describes both the breakdown of geometrical optics and the way in which the  $\mathbf{B}$  vector ceases to 'hug' the surface in an 'adiabatic' manner.

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